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The Complexity of the Hamilton Cycle Problem in Hypergraphs of High Minimum Codegree*

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Abstract

We consider the complexity of the Hamilton cycle decision problem when restricted to k -uniform hypergraphs H of high minimum codegree $\delta(H)$. We show that for tight Hamilton cycles this problem is NP-hard even when restricted to k -uniform hypergraphs H with $\delta(H) \geq \frac{n}{2} - C$, where n is the order of H and C is a constant which depends only on k . This answers a question raised by Karpiński, Ruciński and Szymańska. Additionally we give a polynomial-time algorithm which, for a sufficiently small constant $\varepsilon > 0$, determines whether or not a 4-uniform hypergraph H on n vertices with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ contains a Hamilton 2-cycle. This demonstrates that some looser Hamilton cycles exhibit interestingly different behaviour compared to tight Hamilton cycles. A key part of the proof is a precise characterisation of all 4-uniform hypergraphs H on n vertices with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ which do not contain a Hamilton 2-cycle; this may be of independent interest. As an additional corollary of this characterisation, we obtain an exact Dirac-type bound for the existence of a Hamilton 2-cycle in a large 4-uniform hypergraph.

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1 Introduction

The study of Hamilton cycles in graphs has been a topic of great significance in graph theory, and continues to be well-studied. For example, the Hamilton cycle decision problem (given a graph, determine whether it contains a Hamilton cycle) was one of Karp's celebrated 21 NP-complete problems [9], whilst one very well-known classic result is Dirac's theorem [4], which states that any graph on $n \geq 3$ vertices with minimum degree at least $\frac{n}{2}$ contains a Hamilton cycle.

The problem of generalising these results to the hypergraph setting has been a highly-active area of research over the past several years (see, for example, the recent surveys by Kühn and Osthus [15], Rödl and Ruciński [16] and Zhao [21]). To describe these developments we require the following standard definitions. A k -uniform hypergraph, or k -graph H consists of a set of vertices $V(H)$ and a set of edges $E(H)$, where each edge consists of k vertices. So a 2-graph is a (simple) graph. We say that a k -graph C is an ℓ -cycle if its vertices can be cyclically ordered in such a way that each edge of C consists of k consecutive

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vertices, and so that each edge intersects the subsequent edge in ℓ vertices. This naturally generalises the notion of a cycle in a graph, and is the most commonly-used definition of a hypergraph cycle. However, various other definitions have also been considered, such as a Berge cycle [1]. Note in particular that each edge of an ℓ -cycle k -graph C has $k - \ell$ vertices which were not contained in the previous edge, so the number of vertices of C must be divisible by $k - \ell$. We say that a k -graph H on n vertices contains a *Hamilton ℓ -cycle* if it contains an n -vertex ℓ -cycle as a subgraph; as before, this is only possible if $k - \ell$ divides n . We refer to $(k - 1)$ -cycles as *tight cycles*, and in the same way refer to *tight Hamilton cycles*. Given a k -graph H and a set $S \subseteq V(H)$, the *degree* of S , denoted $\deg_H(S)$ (or $\deg(S)$ when H is clear from the context), is the number of edges of H which contain S as a subset. The *minimum codegree* of H , denoted $\delta(H)$, is the minimum of $\deg(S)$ taken over all sets of $k - 1$ vertices of H , and the *maximum codegree* of H , denoted $\Delta(H)$, is the maximum of $\deg(S)$ taken over all sets of $k - 1$ vertices of H . In the graph case the maximum and minimum codegree are simply the maximum and minimum degree respectively.

An elementary reduction from the graph case demonstrates that for any $k \geq 3$ and $1 \leq \ell \leq k$ the Hamilton ℓ -cycle decision problem (given a k -graph H , determine whether it contains a Hamilton ℓ -cycle) is also NP-complete. For this reason, many authors have asked for conditions on H which render this problem tractable, or which guarantee the existence of a Hamilton ℓ -cycle in H . In particular, since a Hamilton cycle in H cannot exist if H has an isolated vertex, it is natural to study minimum degree conditions on H .

1.1 Dirac-Type Results

The following theorem, whose various cases were proved by Rödl, Ruciński and Szemerédi [17, 18], Kühn and Osthus [14], Keevash, Kühn, Osthus and Mycroft [12], Hàn and Schacht [6], and Kühn, Osthus and Mycroft [13], is an approximate hypergraph analogue of Dirac's theorem; for any k and ℓ it gives the asymptotically best-possible minimum codegree condition which guarantees the existence of a Hamilton ℓ -cycle in a k -graph.

► **Theorem 1.** ([6, 12, 13, 14, 17, 18]) *For any $k \geq 3$, $1 \leq \ell \leq k - 1$ and $\eta > 0$, there exists n_0 such that if $n \geq n_0$ is divisible by $k - \ell$ and H is a k -graph on n vertices with*

$$\delta(H) \geq \begin{cases} \left(\frac{1}{2} + \eta\right)n & \text{if } k - \ell \text{ divides } k, \\ \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + \eta\right)n & \text{otherwise,} \end{cases}$$

then H contains a Hamilton ℓ -cycle.

Simple constructions show that for any k and ℓ this minimum codegree condition is best possible up to the ηn error term. More recently the exact threshold (for large n) has been determined in some cases: for $k = 3, \ell = 2$ by Rödl, Ruciński and Szemerédi [19], for $k = 3, \ell = 1$ by Czygrinow and Molla [2], and for $k \geq 3$ and $\ell < k/2$ by Hàn and Zhao [8]. As part of our work on the question of tractability (described in more detail in the next section), we successfully characterised all 4-graphs H with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ which do not contain a Hamilton cycle. As a straightforward consequence of this, we add to the aforementioned results the exact Dirac-type statement for the previously-open case $k = 4, \ell = 2$.

► **Theorem 2.** *There exists n_0 such that if $n \geq n_0$ is even and H is a 4-graph on n vertices with*

$$\delta(H) \geq \begin{cases} \frac{n}{2} - 2 & \text{if } n \text{ is divisible by } 8, \\ \frac{n}{2} - 1 & \text{otherwise,} \end{cases}$$

then H contains a Hamilton 2-cycle. Moreover, this condition is best-possible for any even $n \geq n_0$.

1.2 Tractability of the Restricted Hamilton Cycle Decision Problem

We now turn to the primary focus of this paper: minimum degree conditions which render the Hamilton cycle decision problem tractable. In the graph case, Dahlhaus, Hajnal and Karpiński [3] showed that for any fixed $\varepsilon > 0$ this problem remains NP-complete when restricted to graphs with minimum degree at least $(1 - \varepsilon)\frac{n}{2}$. More recently, Karpiński, Ruciński and Szymańska [10] showed that for any $k \geq 3$ and any fixed $\varepsilon > 0$ the tight Hamilton cycle decision problem remains NP-complete when restricted to k -graphs with minimum codegree $(1 - \varepsilon)\frac{n}{k}$. They noted that, combined with Theorem 1, this left a ‘hardness gap’ of $[\frac{n}{k}, \frac{n}{2}]$ for which the hardness of the problem remained unknown. We answer this question with the following theorem.

► **Theorem 3.** *For any $k \geq 3$ there exists C such that the tight Hamilton cycle decision problem remains NP-complete when restricted to k -graphs H with $\delta(H) \geq \frac{n}{2} - C$ (where $n = |V(H)|$).*

Assuming that $P \neq NP$, Theorems 1 and 3 together imply that the minimum codegree threshold at which the tight Hamilton cycle decision problem becomes tractable is asymptotically equal to the minimum codegree threshold for the existence of a tight Hamilton cycle, mirroring the situation in the graph case. Interestingly, we can demonstrate that the Hamilton 2-cycle problem exhibits significantly different behaviour; our next theorem shows that there is a linear-size gap between the threshold at which the problem becomes tractable and at which the existence of a cycle is guaranteed.

► **Theorem 4.** *There exist a constant $\varepsilon > 0$ and an algorithm which, given a 4-graph H on n vertices with $\delta(H) \geq \frac{n}{2} - \varepsilon n$, determines in time $O(n^{25})$ whether H contains a Hamilton 2-cycle.*

A slight adaptation of the argument of Karpiński, Ruciński and Szymańska [10] mentioned above shows that for any fixed $\varepsilon > 0$ the Hamilton 2-cycle problem remains NP-complete when restricted to 4-graphs with minimum codegree at least $(1 - \varepsilon)\frac{n}{4}$.

A key result in our proof of Theorem 4, which may be of independent interest, is Theorem 6, which (for sufficiently small ε and large n) precisely characterises all 4-graphs on n vertices which satisfy $\delta(H) \geq \frac{n}{2} - \varepsilon n$ but which do not contain a Hamilton 2-cycle. We prove this result using recently developed techniques of extremal graph theory, in particular the so-called ‘absorbing method’ of Rödl, Ruciński and Szemerédi [17]. Establishing this characterisation is the principal difficulty in the proof of Theorem 4, as then the algorithm for Theorem 4 simply checks whether this characterisation is satisfied. Likewise, Theorem 2 follows from Theorem 6 by a case analysis.

1.3 Discussion

In the light of Theorem 4, it would be very interesting to know which other values of k and ℓ also have the property that there is a linear-size gap between the minimum codegree threshold which renders the k -graph Hamilton ℓ -cycle problem tractable and the minimum codegree threshold under which the problem becomes trivial. Theorem 3 shows that this is not the case when $\ell = k - 1$, whilst a slight adaptation to the arguments of Karpiński,

Ruciński and Szymańska [10] demonstrates that this is also not true if $k - \ell$ does not divide k (in which case the lower degree threshold of Theorem 1 applies); all other cases remain open.

We also note that Theorem 3 demonstrates an interesting difference between the perfect matching problem and tight Hamilton cycle problem in k -graphs. Indeed, while the unrestricted versions of both problems are NP-complete, Keevash, Knox and Mycroft [11] and Han [7] showed that the perfect matching problem can be solved in polynomial time in k -graphs H with $\delta(H) \geq n/k$; complementing a previous result of Szymanska [20], who showed that for any $\varepsilon > 0$ the problem remains NP-complete under the restriction $\delta(H) \geq (\frac{1}{k} - \varepsilon)n$. So, assuming $P \neq NP$, for any $\frac{1}{k} \leq \alpha < \frac{1}{2}$ the two problems lie in distinct complexity classes when restricted to k -graphs with minimum codegree $\delta(H) \geq \alpha n$.

Finally, whilst the constant ε in Theorem 4 is quite small, we conjecture that Theorem 6 (the characterisation of 4-graphs H with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ and no Hamilton 2-cycle) is in fact valid under the weaker condition that $\delta(H) > \frac{n}{3}$. If true, this would imply that Theorem 4 would also hold under this weaker codegree assumption.

1.4 Notation

Given a set V , we write $\binom{V}{k}$ for the set of subsets of V of size k . Also, we write $x \ll y$ (“ x is sufficiently smaller than y ”) to mean that for any $y > 0$ there exists $x_0 > 0$ such that for any $x \leq x_0$ the subsequent statement holds. Similar statements with more variables are defined accordingly.

2 Hamilton 2-Cycles

In this section we outline the proof of Theorem 4. The key to the proof is Theorem 6, which precisely characterises all large 4-graphs H with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ which do not contain a Hamilton 2-cycle. This is presented in Section 2.1. Having established this characterisation, it is fairly straightforward to exhibit a polynomial-time algorithm which tests whether a 4-graph has this property, as shown in Section 2.2. Instead, the difficult part of the proof is to prove Theorem 6; we outline how this is done in Section 2.3. Finally, in Section 2.4 we present the short deduction of Theorem 2 from Theorem 6.

2.1 A Characterisation of Dense 4-graphs with no Hamilton 2-Cycle.

For 4-graphs H , our characterisation considers partitions of $V(H)$ into two parts A and B . Whenever we refer to, for example, ‘a partition (A, B) of $V(H)$ ’, this should be interpreted as meaning a partition of $V(H)$ into two non-empty parts A and B . Given such a partition of $V(H)$, we say that an edge $e \in E(H)$ is *odd* if $|e \cap A|$ is odd, and *even* if $|e \cap A|$ is even. We write H_{even} for the subgraph of H consisting only of even edges of H , and similarly write H_{odd} for the subgraph of H consisting only of odd edges of H . Also, we say that a pair $\{x, y\}$ of distinct vertices of H is a *split* pair if $x \in A$ and $y \in B$ or vice versa, and that $\{x, y\}$ is an *equal* pair if $x, y \in A$ or $x, y \in B$.

We define an ℓ -path in a k -graph analogously to an ℓ -cycle: a k -graph is an ℓ -path if its vertices can be linearly ordered v_1, \dots, v_n such that every edge consists of k consecutive vertices and successive edges intersect in precisely ℓ vertices. As for cycles we refer to $(k-1)$ -paths as *tight paths*. The *length* of an ℓ -path is the number of edges. Given a 4-graph H , we define the *total 2-pathlength* of H to be the maximum sum of lengths of vertex-disjoint 2-paths in H . For example, H having total 2-pathlength 3 could be achieved by 3 disjoint edges (i.e. 2-paths of length 1) in H , or a 2-path of length 3 in H , or two vertex-disjoint

2-paths in H , one of length 1 and one of length 2. Using these definitions we can now give the central definition of our characterisation.

► **Definition 5.** Let H be a 4-graph on n vertices, where n is even. We say that a partition (A, B) of $V(H)$ is *even-good* if at least one of the following statements holds.

- (i) $|A|$ is even or $|A| = |B|$.
- (ii) H contains odd edges e and e' such that either $e \cap e' = \emptyset$ or $e \cap e'$ is a split pair.
- (iii) $|A| = |B| + 2$ and H contains odd edges e and e' with $e \cap e' \in \binom{A}{2}$.
- (iv) $|B| = |A| + 2$ and H contains odd edges e and e' with $e \cap e' \in \binom{B}{2}$.

Now let $m \in \{0, 2, 4, 6\}$ and $d \in \{0, 2\}$ be such that $m \equiv n \pmod{8}$ and $d \equiv |A| - |B| \pmod{4}$. Then we say that (A, B) is *odd-good* if at least one of the following statements holds.

- (v) $(m, d) \in \{(0, 0), (4, 2)\}$.
- (vi) $(m, d) \in \{(2, 2), (6, 0)\}$ and H contains an even edge.
- (vii) $(m, d) \in \{(4, 0), (0, 2)\}$ and H_{even} has total 2-pathlength at least two.
- (viii) $(m, d) \in \{(6, 2), (2, 0)\}$ and either there is an edge $e \in E(H)$ with $|e \cap A| = |e \cap B| = 2$ or H_{even} has total 2-pathlength at least three.

A key observation is that if (A, B) is a partition of $V(H)$ which is not even-good, then there exists a set X of at most four vertices of H such that every odd edge of H intersects X . Indeed, if H contains an odd edge e , then we may take $X = e$, and otherwise we may take $X = \emptyset$. Similarly, by choosing X to be the vertices of at most two disjoint even edges, or of a 2-path of length two in H_{even} , we find that if (A, B) is a partition of $V(H)$ which is not odd-good, then there exists a set X of at most 8 vertices of H such that every even edge of H intersects X .

We now give our characterisation of 4-graphs of high minimum codegree with no Hamilton 2-cycle. Recall for this that any 2-cycle 4-graph has an even number of vertices.

► **Theorem 6.** *There exist $\varepsilon, n_0 > 0$ such that the following statement holds for any even $n \geq n_0$. Let H be a 4-graph on n vertices with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$. Then H admits a Hamilton 2-cycle if and only if every partition (A, B) of $V(H)$ is both even-good and odd-good.*

2.2 The Algorithm.

Our polynomial-time algorithm for determining the existence of a Hamilton 2-cycle in a 4-graph of high codegree makes use of a special case of a result of Keevash, Knox and Mycroft [11]. This result allows us to efficiently list all partitions (A, B) of $V(H)$ with no odd edges, or all partitions with no even edges.

► **Lemma 7.** *([11], special case of Lemma 2.2) Let H be a 4-graph on n vertices with $\delta(H) > \frac{n}{3}$, and let $x \in \{\text{even}, \text{odd}\}$. Then there are at most 64 partitions (A, B) of $V(H)$ for which no edge of H has parity x with respect to (A, B) . Moreover, there exists an algorithm $\text{ListPartitions}(H, x)$ with running time $O(n^5)$ which, given H and x , returns all such partitions.*

We now present an algorithm, $\text{Procedure GoodPartition}(H, x)$, which determines whether or not there exists an even-good/odd-good partition (A, B) for a 4-graph H . Note that, given a 4-graph H and a partition (A, B) of $V(H)$, the truth of the statements ‘ (A, B) is odd-good’ and ‘ (A, B) is even-good’ depend only on the values of n and $|A|$ and whether or not H_{odd} or H_{even} contain certain subgraphs with at most 12 vertices. It follows that the validity of these statements (and therefore the condition of the ‘if’ statement in $\text{Procedure GoodPartition}$) can be tested in time $O(n^{12})$.

Procedure GoodPartition(H, x)

Data: A 4-graph H with vertex set V and a parity $x \in \{\text{even}, \text{odd}\}$.

Result: Determines if there is a partition (A, B) of V which is not x -good.

```

for each set  $X \subseteq V(H)$  with  $|X| = 8$  do
    Let  $V' = V \setminus X$  and  $H' = H[V']$ .
    Run Procedure ListPartitions( $H', x$ ) to obtain all partitions  $(A', B')$  of  $V'$  with no
    edges not of parity  $x$ .
    for each such partition  $(A', B')$  do
        for each partition  $(A, B)$  of  $V$  with  $A' \subseteq A$  and  $B' \subseteq B$  do
            if  $(A, B)$  is not  $x$ -good then
                State ' $(A, B)$  is not  $x$ -good', and terminate.
        State 'Every partition is  $x$ -good', and terminate.

```

► **Proposition 8.** Let H be a 4-graph on n vertices with $\delta(H) > \frac{n}{3}$, where n is even, and fix a parity $x \in \{\text{even}, \text{odd}\}$. Then Procedure GoodPartition(H, x) will correctly determine whether there exists a partition (A, B) of $V(H)$ which is not x -good, with running time $O(n^{25})$.

Proof. We first establish correctness of the algorithm; for this, fix H and x as in the proposition statement. Clearly, if every partition (A, B) of $V := V(H)$ is x -good, then GoodPartition(H, x) will output this fact. So suppose that some partition (A, B) of V is not x -good. As noted following Definition 5, we may then choose a set X of at most 8 vertices of H which is intersected by every edge of H which does not have parity x . This means that when GoodPartition(H, x) considers this set X , ListPartitions will return the partition (A', B') where $A' = A \setminus X$ and $B' = B \setminus X$, and at this point GoodPartition(H, even) will return that (A, B) is not x -good, as required.

Finally we consider the running time of the algorithm. For this note that there are $\binom{n}{8}$ choices for X in the outside 'for loop', and for each of these Procedure ListPartitions(H', x) runs in time $O(n^5)$. The inside 'for loops' then range over sets of size at most 64 (by Lemma 7) and $2^8 = 256$ respectively. Finally, as noted above we may test whether a partition (A, B) is x -good in time $O(n^{12})$; together these bounds combine to give the claimed running time. ◀

Proof of Theorem 4. Let n_0 be sufficiently large and $\varepsilon > 0$ sufficiently small for Theorem 6 to apply. Given a 4-graph H on n vertices with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$, we apply the following algorithm. Firstly, if n is odd, then there can be no Hamilton 2-cycle in H , so we output this fact and terminate. Secondly, if $n < n_0$, then we use a brute-force approach, testing each of the at most $n_0!$ orderings of $V(H)$ in turn to determine whether it yields a Hamilton 2-cycle in H . We then output the appropriate answer and terminate. Finally, if $n \geq n_0$ is even, then we first run Procedure GoodPartition(H, even), and then run Procedure GoodPartition(H, odd). If either of these procedures yields a partition (A, B) of $V(H)$ which is not even-good or which is not odd-good, then we return that there is no Hamilton 2-cycle in H , otherwise we return that there is such a cycle. Note that in the first two cases this algorithm runs in constant time, whilst in the final case it runs in time $O(n^{25})$ by Proposition 8. Moreover Theorem 6 ensures that this algorithm will always output the correct answer. ◀

2.3 Proof of Theorem 6.

We begin by establishing the forward implication of Theorem 6, expressed in the following proposition. In fact, the minimum codegree condition on H is not required for this direction.

► **Proposition 9.** If H is a 4-graph which contains a Hamilton 2-cycle, then every partition (A, B) of $V(H)$ is both even-good and odd-good.

Proof. Let n be the order of H , let $C = (v_1, v_2, \dots, v_n)$ be a Hamilton 2-cycle in H and let (A, B) be a partition of $V(H)$. Write $P_i = \{v_{2i-1}, v_{2i}\}$ for each $1 \leq i \leq \frac{n}{2}$, so the edges of C are $e_i := P_i \cup P_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ (with addition taken modulo $\frac{n}{2}$). The key observation is that e_i is even if P_i and P_{i+1} are both split pairs or both equal pairs, and odd otherwise.

We first show that (A, B) is even-good. This holds by (ii) if H contains two disjoint odd edges, so we may assume without loss of generality that all edges of H other than e_1 and $e_{n/2}$ are even. It follows that the pairs $P_2, P_3, \dots, P_{n/2}$ are either all split pairs or all equal pairs. In the former case, if P_1 is a split pair then $|A| = |B|$, so (i) holds, whilst if $P_1 \subseteq A$ then (iii) holds, and if $P_1 \subseteq B$ then (iv) holds. In the latter case, if P_1 is an equal pair then $|A|$ is even, so (i) holds, whilst if P_1 is a split pair then (ii) holds. So in all cases we find that (A, B) is even-good.

To show that (A, B) is odd-good, suppose first that 4 does not divide n , and note that by our key observation the number of even edges in C must then be odd. If C contains three or more even edges or an edge with precisely two vertices in A , then (A, B) is odd-good by (vi) and (viii), so we may assume without loss of generality that $e_{n/2}$ is the unique even edge in C and that $e_{n/2} \subseteq A$ or $e_{n/2} \subseteq B$. It follows that $P_1, P_3, \dots, P_{n/2}$ are equal pairs and the remaining pairs are split, so $|A| - |B| \equiv 2\lceil \frac{n}{4} \rceil \pmod{4}$. We must therefore have $(m, d) \in \{(2, 2), (6, 0)\}$, and (A, B) is odd-good by (vi). On the other hand, if 4 divides n , then by our key observation the number of even edges in C is even. If this number is at least two then (A, B) is odd-good by (v) and (vii). If instead every edge of C is odd, then exactly $\frac{n}{4}$ of the pairs P_i are equal pairs, so $|A| - |B| \equiv \frac{n}{2} \pmod{4}$, and C is odd-good by (v). ◀

To prove Theorem 6 it therefore suffices to prove the backwards implication. Our approach for this is motivated by the observation that if H is a 4-graph and (A, B) is a partition of $V(H)$ which is not odd-good, then H must have very few even edges. Likewise, if (A, B) is not even-good, then H has very few odd edges. We therefore consider three cases for H : two ‘near-extremal’ cases, in which $V(H)$ admits a partition (A, B) with few even edges or with few odd edges, and a ‘non-extremal’ case, in which there is no such partition. In the ‘non-extremal case’ we proceed by the so-called ‘absorbing’ method, introduced by Rödl, Ruciński and Szemerédi [19], in which we rely heavily on the fact that H is not ‘near-extremal’. On the other hand, in the ‘near-extremal’ cases we have significant information about the structure of H (specifically that there is a partition of $V(H)$ with few even/odd edges). Making essential use of this structural information, we proceed by *ad hoc* methods to construct a Hamilton 2-cycle in H .

The following definition formalises our two notions of ‘near-extremal’.

► **Definition 10.** Let $c_1, c_2 > 0$ and let H be a 4-graph on n vertices.

- (a) We say that H is c_1 -even-extremal if there exists a partition (A, B) of $V(H)$ such that $(\frac{1}{2} - c_1)n \leq |A| \leq (\frac{1}{2} + c_1)n$ and H contains at most $c_1 \binom{n}{4}$ odd edges.
- (b) We say that H is c_2 -odd-extremal, if there exists a partition (A, B) of $V(H)$ such that $(\frac{1}{2} - c_2)n \leq |A| \leq (\frac{1}{2} + c_2)n$ and H contains at most $c_2 \binom{n}{4}$ even edges.

2.3.1 Non-Extremal 4-Graphs

As described above, in the case when H is not near-extremal, we proceed by the ‘absorbing’ method of Rödl, Ruciński and Szemerédi [19]. To do this we establish three key lemmas. The first of these is a ‘connecting lemma’, which shows that since H is not even-extremal, we can find a constant-length 2-path connecting any two disjoint pairs of vertices. For this, we say that the *ends* of a 2-path 4-graph (v_1, \dots, v_n) are the pairs $\{v_1, v_2\}$ and $\{v_{n-1}, v_n\}$.

► **Lemma 11** (Connecting lemma). *Suppose that $\frac{1}{n} \ll \varepsilon \ll c$ and that H is a 4-graph on n vertices with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ which is not c -even-extremal. Then for every two disjoint pairs $\{a_1, a_2\}, \{b_1, b_2\} \in \binom{V}{2}$ there is a 2-path of length at most 3 whose ends are $\{a_1, a_2\}$ and $\{b_1, b_2\}$.*

Loosely speaking, our proof of Lemma 11 supposes that we have pairs $\{a_1, a_2\}$ and $\{b_1, b_2\}$ for which no such 2-path exists. It follows that there is no pair $\{x, y\} \in \binom{V(H)}{2}$ for which $\{a_1, a_2, x, y\}$ and $\{b_1, b_2, x, y\}$ are both edges of H . Combined with the minimum codegree condition of H this yields significant structural information on H , which we use to deduce that H must be c -even-extremal and so prove the lemma.

The second key lemma is an ‘absorbing lemma’, which shows that since H is neither even-extremal nor odd-extremal, we can find a short 2-path in H which can ‘absorb’ most small collections of pairs of H .

► **Lemma 12.** (Absorbing lemma) *Suppose that $\frac{1}{n} \ll \varepsilon \ll \rho \ll \beta \ll \lambda \ll c, \mu$. Let H be a 4-graph on n vertices with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ which is neither c -even-extremal nor c -odd-extremal. Then there is a 2-path P in H and a graph G on $V(H)$ with the following properties.*

- (i) P has at most μn vertices.
- (ii) Every vertex of $V(H) \setminus V(P)$ lies in at least $(1 - \lambda)n$ edges of G .
- (iii) For any $q \leq \rho n$ and any q disjoint edges e_1, \dots, e_q of G which do not intersect P there is a 2-path P^* in H with the same ends as P such that $V(P^*) = V(P) \cup \bigcup_{j=1}^q e_j$.

Loosely speaking, to prove Lemma 12, we first show that provided H is not c -odd-extremal, for almost every pair $\{x, y\} \in \binom{V}{2}$ there are many 2-paths Q of length 3 which can ‘absorb’ $\{x, y\}$, in the sense that there is a 2-path Q^* with vertex set $V(Q) \cup \{x, y\}$ and with the same ends as Q . We take G to be the graph of such pairs. We then randomly select a linear number of 2-paths of length 3 and use Lemma 11 to connect these 2-paths into a single short 2-path P (this is where we require that H is not c -even-extremal). Next we extend P to include the small number of vertices which lie in fewer than $(1 - \lambda)n$ edges of G , so that (ii) holds. Finally, we show that given any set of edges e_1, \dots, e_q of G as in (iii), we can match these edges to the randomly chosen paths Q , and absorb each edge into the corresponding path to obtain P^* .

Our final key lemma is a ‘path cover lemma’, which states that we can cover almost all vertices of H by a constant number of vertex-disjoint 2-paths. In fact, we do not actually need the requirement that H is not near-extremal, and can simply cite a result of Kühn, Mycroft and Osthus [13].

► **Lemma 13** (Path cover lemma [13]). *Suppose that $\frac{1}{n} \ll \frac{1}{D} \ll \gamma \ll \eta$ and that H is a 4-graph on n vertices with $\delta(H) \geq (\frac{1}{4} + \eta)n$. Then H contains a set of at most D vertex-disjoint 2-paths covering all but at most γn vertices of H .*

For non-extremal 4-graphs H , combining these three lemmas proves the reverse implication of Theorem 6, which we express in the following lemma.

► **Lemma 14.** *Suppose that $\frac{1}{n} \ll \varepsilon \ll c$ and that n is even, and let H be a 4-graph of order n with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$. If H is neither c -odd-extremal nor c -even-extremal, then H contains a Hamilton 2-cycle.*

Proof sketch. Introduce constants with $1/n \ll 1/D, \varepsilon \ll \gamma \ll \rho \ll \beta \ll \lambda \ll c, \mu \ll 1$, and apply Lemma 12 to obtain an absorbing 2-path P_0 in H and a graph G on $V(H)$ with the stated properties. Let $V := V(H)$ and $U := V(P_0)$, and now choose uniformly at random a set $R \subseteq V \setminus U$ of size ρn . Next, apply Lemma 13 (with, say, $\eta = 1/10$) to obtain at most D vertex-disjoint 2-paths P_1, \dots, P_q in $H[V \setminus (U \cup R)]$ covering all but at most γn vertices. By q applications of Lemma 11 we can find vertex-disjoint 2-paths Q_0, Q_1, \dots, Q_q , each of length at most 3, such that Q_0 connects the end of P_0 to the start of P_1 , Q_1 connects the end of P_1 to the start of P_2 , and so forth, with Q_q connecting the end of P_q to the start of P_0 . Moreover, all vertices of Q_i except those in the end of P_i or the start of P_{i+1} should be taken from R . (The random choice of R ensures that the conditions of Lemma 11 are satisfied for each application.) This yields a 2-cycle $C = P_0 Q_0 P_1 Q_1 P_2 \dots P_q Q_q$ in H covering all vertices except the at most γn vertices not covered by P_1, \dots, P_q and between $\rho n - 3D$ and ρn unused vertices of R . So $X := V \setminus V(C)$ has size $\rho n - 3D \leq |X| \leq \rho n + \gamma n$. Furthermore, $|X|$ is even since n and $|V(C)|$ are both even, and our random choice of R ensures that every vertex $x \in X$ has $\deg_{G[X]}(x) \geq |X|/2$. So there is a perfect matching $e_1, \dots, e_{|X|/2}$ in $G[X]$; since $|X|/2 \leq \rho n$ we may ‘absorb’ X into P_0 to obtain a 2-path P^* . Replacing P_0 by P^* in C gives a Hamilton 2-cycle in H . ◀

2.3.2 Extremal 4-Graphs

Having dealt with the ‘non-extremal’ case, it remains to deal with the two ‘near-extremal’ cases by proving the following two lemmas via an extremal case.

► **Lemma 15.** *Suppose that $\frac{1}{n} \ll \varepsilon, c \ll 1$ and that n is even, and let H be a 4-graph of order n with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$. If H is c -even-extremal and every partition of $V(H)$ into two parts A and B is even-good, then H contains a Hamilton 2-cycle.*

► **Lemma 16.** *Suppose that $\frac{1}{n} \ll \varepsilon, c \ll 1$ and that n is even, and let H be a 4-graph of order n with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$. If H is c -odd-extremal and every partition of $V(H)$ into two parts A and B is odd-good, then H contains a Hamilton 2-cycle.*

Proof sketch. As is typical of this type of argument, each lemma is proved by a long and detailed extremal case analysis, and so we limit ourselves here to a brief outline of the argument for Lemma 15 (the outline for Lemma 16 is similar with ‘even’ and ‘odd’ reversed). Let (A', B') be a partition of $V(H)$ witnessing that H is c -even-extremal. We first observe that the bound on $\delta(H)$ implies that H has density at least $(\frac{1}{2} - \varepsilon)$. Combined with the fact that H has few odd edges, this implies that almost every set $S \subseteq V(H)$ for which $|A' \cap S|$ is even is an edge of H . However, it is possible that a small number of vertices may lie in very few even edges, so we begin by ‘tidying up’ the partition: we move a few vertices of H from one side to the other to ensure that, for instance, every vertex of H lies in many even edges. Let (A, B) be the tidied partition. By assumption this partition (A, B) is even-good, and this fact yields some structure in H with respect to this partition (precisely what structure depends on the values of n and $|A|$). For example, we might obtain two disjoint odd edges in H . We then form a short 2-path P from the given structure to satisfy the desired parity conditions, and then (using even edges only) extend P to a Hamilton 2-cycle in H . ◀

Proof of Theorem 6. Fix a constant c small enough for Lemmas 15 and 16. Having done so, choose ε sufficiently small for us to apply Lemma 14 with this choice of c , and n_0 sufficiently

large that we may apply Lemmas 14, 15 and 16 with these choices of c and ε and any even $n \geq n_0$. Let H be a 4-graph on n vertices with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$, and suppose that every partition (A, B) of $V(H)$ is both even-good and odd-good. If H is either c -even-extremal or c -odd-extremal then H contains a Hamilton 2-cycle by Lemma 15 or 16 respectively. On the other hand, if H is neither c -odd-extremal nor c -even-extremal then H contains a Hamilton 2-cycle by Lemma 14. This completes the proof of the backwards implication of Theorem 6; the proof of the forwards implication was Proposition 9. \blacktriangleleft

2.4 Proof of Theorem 2

To conclude this section, we show how Theorem 2 can be deduced from Theorem 6. We begin by justifying the claim that the degree bound of Theorem 2 is best-possible. To see this, fix an even integer $n \geq 6$, and construct a 4-graph H^* as follows. Let A and B be disjoint sets with $|A \cup B| = n$ such that $|A| = \frac{n}{2} - 1$ if 8 divides n and $|A| = \frac{n}{2}$ otherwise. Then the vertex set of H^* is $A \cup B$, and the edges of H^* are all sets $e \in \binom{A \cup B}{4}$ such that $|e \cap A|$ is odd. Then it is easily checked that $\delta(H^*) = \frac{n}{2} - 3$ if 8 divides n and $\frac{n}{2} - 2$ otherwise. Moreover, since H^* has no even edges, our choice of size of A implies that the partition (A, B) of $V(H^*)$ is not odd-good. By Theorem 6 we conclude that there is no Hamilton 2-cycle in H^* .

Proof of Theorem 2. Choose ε, n_0 as in Theorem 6. Let $n \geq n_0$ be even and large enough that $\frac{n}{2} - 2 \geq (\frac{1}{2} - \varepsilon)n$, and let H be a 4-graph on n vertices which satisfies the minimum codegree condition of Theorem 2. Also let (A, B) be a partition of $V(H)$, and assume without loss of generality that $|A| \leq \frac{n}{2}$. By Theorem 6 it suffices to prove that (A, B) is even-good and odd-good. For this, note that if 8 divides n and $|A| = \frac{n}{2}$ then (A, B) is even-good by (i) and odd-good by (v). So we may assume that if 8 divides n then $|A| \leq \frac{n}{2} - 1$ and $\delta(H) \geq \frac{n}{2} - 2$, whilst otherwise we have $|A| \leq \frac{n}{2}$ and $\delta(H) \geq \frac{n}{2} - 1$. Either way, we must have $\delta(H) \geq |A| - 1$. Also, for any distinct $x, y, z \in V(H)$, let $N_B(x, y, z)$ denote the set of vertices $w \in B$ such that $\{x, y, z, w\} \in E(H)$.

To see that (A, B) must be even-good, arbitrarily choose vertices $x_1, x_2, y_1, y_2, z_1, z_2 \in A$. Then $|N_B(x_1, y_1, z_1)|, |N_B(x_2, y_2, z_2)| \geq \delta(H) - (|A| - 3) \geq 2$, so we may choose distinct $w_1, w_2 \in B$ with $w_1 \in N_B(x_1, y_1, z_1)$ and $w_2 \in N_B(x_2, y_2, z_2)$. The sets $\{x_1, y_1, z_1, w_1\}$ and $\{x_2, y_2, z_2, w_2\}$ are then disjoint odd edges of H , so (A, B) is even-good by (ii).

We next show that (A, B) is also odd-good. For this, arbitrarily choose distinct vertices $a_1, a_2, \dots, a_9, a'_1, \dots, a'_9 \in A$ and $b_1, \dots, b_9 \in B$. For any $1 \leq i, j \leq 9$ we have $|N_B(a_i, a'_i, b_j)| \geq \delta(H) - (|A| - 2) \geq 1$, so there must be $b_j^i \in B$ such that $\{a_i, a'_i, b_j, b_j^i\}$ is an (even) edge of H . If for each $1 \leq j \leq 9$ the vertices b_j^i for $1 \leq i \leq 9$ are all distinct, then there is no set $X \subseteq V(H)$ with $|X| \leq 8$ which intersects every even edge of H . However, as observed immediately after Definition 5, such a set X must exist if (A, B) is not odd-good. We may therefore assume that $b_j^{i'} = b_j^i$ for some $1 \leq i, i', j \leq 9$ with $i \neq i'$. It follows that $\{a_i, a'_i, b_j, b_j^i\}$ is an even edge of H with exactly two vertices in A , whilst $(a_i, a'_i, b_j, b_j^i, a_{i'}, a'_{i'})$ is a 2-path of length 2 in H_{even} . So (A, B) is odd-good by (v), (vi), (vii) or (viii), according to the value of n modulo 8. \blacktriangleleft

3 Tight Hamilton Cycles

Our aim in this section is to explain the principal ideas of the proof of Theorem 3, which proceeds by a series of reductions. We begin with a full proof of the case $k = 3$, in which case we proceed from a theorem of Garey, Johnson and Stockmeyer [5], who proved that the

Hamilton cycle problem remains NP-complete when restricted to subcubic graphs (we say that a graph G is *subcubic* if G has maximum degree $\Delta(G) \leq 3$). The following proposition is an immediate corollary of that theorem.

► **Proposition 17** ([5]). The problem of determining whether a subcubic graph admits a Hamilton path is NP-complete.

The next lemma is the $k = 3$ case of Theorem 3, which holds with $C = 9$.

► **Lemma 18.** *The 3-graph tight Hamilton cycle decision problem is NP-complete even when restricted to 3-graphs H on m vertices with $\delta(H) \geq \frac{m}{2} - 9$.*

Proof. Let G be a subcubic graph on n vertices, and write $X := V(G)$. Assume for simplicity that n is even (a very similar argument handles the case where n is odd). Fix disjoint sets A and B with $|A| = \frac{3n}{2}$ and $|B| = \frac{3n}{2} + 1$ such that $X \subseteq A$, and define a 3-graph H with vertex set $A \cup B$ whose edges are

- (i) all sets $e \in \binom{A \cup B}{3}$ with $|A \cap e| \leq 1$,
- (ii) all sets $e \in \binom{A \cup B}{3}$ with $|A \cap e| = 2$ and $A \cap e \in E(G)$ (note in particular that this requires that $A \cap e \subseteq X$), and
- (iii) all sets $e \in \binom{A}{3}$ for which no $e' \in E(G)$ satisfies $e' \subseteq e$.

Observe first that H has $m := 3n + 1$ vertices and minimum codegree $\delta(H) \geq \frac{m}{2} - 9$. To see this, let x and y be distinct vertices of H . If either $x \in B$ or $y \in B$ then $\{x, y, z\}$ is an edge of H for any $z \in B \setminus \{x, y\}$, so $\deg_H(\{x, y\}) \geq |B| - 2 = \frac{3n}{2} - 1$. Exactly the same applies if $x, y \in A$ and $xy \in E(G)$. Finally, if $x, y \in A$ and $xy \notin E(G)$, then $\{x, y, z\}$ is an edge of H for any $z \in A \setminus \{x, y\}$ except for those z such that $xz \in E(G)$ or $yz \in E(G)$. So $\deg_H(\{x, y\}) \geq |A| - 2 - \deg_G(x) - \deg_G(y)$; since G is subcubic this gives $\deg_H(\{x, y\}) \geq \frac{3n}{2} - 8 \geq \frac{m}{2} - 9$, as claimed.

We claim that H contains a tight Hamilton cycle if and only if G contains a Hamilton path. To see this, first suppose that G contains a Hamilton path (x_1, \dots, x_n) . Enumerate the vertices of $A \setminus X$ and B as $a_1, a_2, \dots, a_{n/2}$ and $b_1, b_2, \dots, b_{3n/2+1}$ respectively. Then

$$(x_1, x_2, b_1, x_3, x_4, b_2, \dots, x_{n-1}, x_n, b_{\frac{n}{2}}, b_{\frac{n}{2}+1}, a_1, b_{\frac{n}{2}+2}, b_{\frac{n}{2}+3}, a_2, \dots, a_{\frac{n}{2}}, b_{\frac{3n}{2}}, b_{\frac{3n}{2}+1})$$

is a tight Hamilton cycle in H .

Now suppose instead that H contains a tight Hamilton cycle C . Note that our construction of H ensures that there are no edges $e, e' \in E(H)$ with $|e \cap A| = 3$, $|e' \cap A| = 2$ and $|e \cap e'| = 2$. Since every edge of C intersects the subsequent edge of C in precisely two vertices, and $B \neq \emptyset$, it follows that C cannot contain any edge e with $|e \cap A| = 3$. So there are at least $\frac{n}{2}$ vertices $a \in X$ which are succeeded in C by a vertex of B . Now let A_1 be the set of vertices of X for which the subsequent vertex of A on C is in X and A_2 be the set of vertices of X for which the subsequent vertex of A on C is in $A \setminus X$. Also let $A_3 := A \setminus X$, so A is the disjoint union of A_1 , A_2 and A_3 . By construction of H , any vertex of $A \setminus X$ must be preceded in C by two vertices of B and succeeded in C by two vertices of B ; it follows that any vertex of $A_2 \cup A_3$ is succeeded in C by two vertices of B , and so we obtain

$$|B| \geq (\frac{n}{2} - |A_2|) + 2(|A_2| + |A_3|) = \frac{n}{2} + |A_2| + 2|A_3| = \frac{3n}{2} + |A_2|.$$

Since $A \setminus X$ is non-empty, we must have $|A_2| \geq 1$. Combined with the fact that $|B| = \frac{3n}{2} + 1$ this implies that $|A_2| = 1$, and all inequalities are in fact equalities. So precisely one vertex of X is succeeded in C by two vertices of B , $\frac{n}{2} - 1$ vertices of X are succeeded

by one vertex of B , and the remaining $\frac{n}{2}$ vertices of X are succeeded by a vertex of A (which must therefore be in X). This implies that C contains a tight Hamilton path of the form $(x_1, x_2, b_1, x_3, x_4, b_2, \dots, b_{n/2-1}, x_{n-1}, x_n)$, where $X = \{x_1, \dots, x_n\}$ and $b_i \in B$ for $1 \leq i \leq \frac{n}{2} - 1$. By our construction of H it follows that (x_1, x_2, \dots, x_n) is a Hamilton path in G .

Altogether, this shows that any instance of the Hamilton cycle problem for subcubic graphs can be reduced to a single instance of the problem of finding a tight Hamilton cycle in a 3-graph on m vertices with $\delta(H) \geq \frac{m}{2} - 9$, where $m = 3n + 1$. Together with Proposition 17, this proves the lemma. \blacktriangleleft

We conclude by outlining the steps we use to prove Theorem 3 in full generality, using the following notation. For a function $f(n)$, we write $\text{HC}(k, f(n))$ (respectively $\text{HP}(k, f(n))$) to denote the k -graph tight Hamilton cycle (respectively Hamilton path) decision problem restricted to k -graphs H on n vertices with minimum codegree $\delta(H) \geq f(n)$. On the other hand, for an integer D , we write $\overline{\text{HC}}(k, D)$ (respectively $\overline{\text{HP}}(k, D)$) to denote the k -graph tight Hamilton cycle (respectively Hamilton path) decision problem restricted to k -graphs H with maximum codegree $\delta(H) \leq D$. So, for example, Proposition 17 states that $\overline{\text{HP}}(2, 3)$ is NP-complete, whilst Lemma 18 states that $\text{HC}(3, \frac{n}{2} - 9)$ is NP-complete. We prove Theorem 3 by exhibiting the following polynomial-time reductions.

- (i) For any $k \geq 2$ and D we give polynomial-time reductions from $\overline{\text{HC}}(k, D)$ to $\overline{\text{HP}}(k, D)$ and from $\overline{\text{HP}}(k, D)$ and $\overline{\text{HC}}(k, D)$. These reductions are elementary and permit us the convenience of treating the tight Hamilton cycle and tight Hamilton path problems in graphs of low maximum codegree as being interchangeable.
- (ii) For any $k \geq 2$ we give polynomial-time reductions from $\overline{\text{HC}}(k, D)$ to $\overline{\text{HC}}(2k-1, 2D)$ and from $\overline{\text{HC}}(k, D)$ to $\overline{\text{HC}}(2k, D)$. In each case, given a k -graph H on a vertex set V , we take copies H_1 and H_2 of H with disjoint vertex sets V_1 and V_2 . For the former reduction we define a $(2k-1)$ -graph H^* on $V_1 \cup V_2$ whose edges are those $(2k-1)$ -tuples which consist of an edge e_1 from H_1 and the copies in H_2 of $k-1$ vertices of e_1 , or the same with the roles of H_1 and H_2 reversed. Likewise, for the latter reduction we define a $2k$ -graph H^* on $V_1 \cup V_2$ whose edges are those $2k$ -tuples $e_1 \cup e_2$ where e_1 is an edge of H_1 , e_2 is an edge of H_2 , and e_2 contains the copies of at least $k-1$ vertices of e_1 . In either case it is not too hard to show that H^* contains a tight Hamilton cycle if and only if H does, and that $\Delta(H^*) \leq 2\Delta(H)$ in one case and $\Delta(H^*) \leq \Delta(H)$ in the other.
- (iii) Finally, for any $k \geq 2$ we present a polynomial-time reduction from $\overline{\text{HP}}(k, D)$ to $\text{HC}(2k-1, \lfloor \frac{n}{2} \rfloor - k(D+1))$ and from $\overline{\text{HC}}(k, D)$ to $\text{HC}(2k, \frac{n}{2} - k(D+1))$. These are similar to the reduction given in the proof of Lemma 18, except that G is now a k -graph with $\Delta(G) \leq D$, and H is a $(2k-1)$ -graph or $2k$ -graph (according to which reduction we are presenting).

By induction on k , with Proposition 17 as the base case, the reductions of (i) and (ii) combine to prove the following theorem, which can be seen as a generalisation to k -graphs of the aforementioned theorem of Garey, Johnson and Stockmeyer.

► Theorem 19. *For every $k \geq 2$ there exists D such that $\overline{\text{HC}}(k, D)$ and $\overline{\text{HP}}(k, D)$ are NP-complete.*

Theorem 3 follows immediately from Theorem 19 and the reductions of (iii).

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